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Note

A Note on the Leap Frog Schemes in any Number of Space Variables

1. *Introduction*

Abarbanel and Gottlieb [1] examined the Leap Frog scheme and a modified form of it for the hyperbolic system of equations

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{u})}{\partial y} = 0.$$

The standard and modified schemes generalize respectively, for the hyperbolic system of equations

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{p=1}^{m} \frac{\partial \mathbf{F}_p(\mathbf{u})}{\partial x_p} = 0 \tag{1}$$

in $m \geqslant 1$ space variables x_p, $p = 1, 2,..., m$, to

$$\mathbf{U}_{\mathbf{j}}^{n+1} = \mathbf{U}_{\mathbf{j}}^{n-1} - \sum_{p=1}^{m} \frac{\Delta t}{\Delta x_p} \delta_p \mathbf{F}_{p;\mathbf{j}}^{n} \tag{2}$$

and

$$\mathbf{U}_{\mathbf{j}}^{n+1} = \mathbf{U}_{\mathbf{j}}^{n-1} - \sum_{p=1}^{m} \frac{\Delta t}{\Delta x_p} \delta_{\nu;p} \mathbf{F}_{p;\mathbf{j}}^{n}. \tag{3}$$

A third scheme

$$\mathbf{U}_{\mathbf{j}}^{n+1} = \mathbf{U}_{\mathbf{j}}^{n-1} - \frac{1}{2} \sum_{p=1}^{m} \frac{\Delta t}{\Delta x_p} (\delta_p \mathbf{F}_{p;\mathbf{j}}^{n} + \delta_{\nu;p} \mathbf{F}_{p;\mathbf{j}}^{n}) \tag{4}$$

is also considered, this being the average of (2) and (3). In these

$$\delta_p \mathbf{V}_{\mathbf{j}} = \mathbf{V}_{\mathbf{j}+\mathbf{e}_p} - \mathbf{V}_{\mathbf{j}-\mathbf{e}_p}, \qquad \delta_{\nu;p} \mathbf{V}_{\mathbf{j}} = \frac{1}{2^{m-1}} \Big(\sum_{\mathbf{i}\in\Omega_1} \mathbf{V}_{\mathbf{i}} - \sum_{\mathbf{i}\in\Omega_2} \mathbf{V}_{\mathbf{i}} \Big),$$

where

$$\Omega_1 = \{\mathbf{i}: |\mathbf{i}_k - \mathbf{j}_k| = \mathbf{e}_k, k = 1, 2,..., m, k \neq p, \mathbf{i}_p = \mathbf{j}_p + 1\},$$
$$\Omega_2 = \{\mathbf{i}: |\mathbf{i}_k - \mathbf{j}_k| = \mathbf{e}_k, k = 1, 2,..., m, k \neq p, \mathbf{i}_p = \mathbf{j}_p - 1\},$$

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with $\mathbf{j} = (j_1, j_2, \ldots, j_m)$ the vector of suffices and $\mathbf{e}_p$ a unit vector in m space with pth entry unity.

2. *Stability Analysis*

Linearizing (2), (3), and (4) by putting $\mathbf{F}_p(\mathbf{u})_{x_p} = A_p \mathbf{U}_{x_p}$, A_p constant, and applying standard Fourier analysis leads to an amplification matrix for each of the schemes (2), (3), and (4).

By the use of characteristic theory, as in [2], it follows that the speed of propagation c of waves in a specified direction is an eigenvalue of $E = \sum_{p=1}^{m} A_p l_p$, where l_p for $p = 1, 2, \ldots, m$ are the direction cosines of the characteristic surface.

The amplification matrices for (2), (3), and (4) can all be written as a function of the matrix E, this enabling the eigenvalues to be expressed as a function of c. Bounding the absolute value of these eigenvalues by unity leads for (2), (3), and (4) respectively to the stability results

$$\frac{\Delta t}{\Delta x} \leqslant \frac{1}{|(\sum_{p=1}^{m} \sin^2 \alpha_p)^{1/2} c|}, \tag{5}$$

$$\frac{\Delta t}{\Delta x} \leqslant \frac{1}{|(\sum_{p=1}^{m} \sin^2 \alpha_p \prod_{q=1, q\neq p}^{m} \cos^2 \alpha_q)^{1/2} c|}, \tag{6}$$

and

$$\frac{\Delta t}{\Delta x} \leqslant \frac{2}{|(\sum_{p=1}^{m} \sin^2 \alpha_p [1 + \prod_{q=1, q\neq p}^{m} \cos^2 \alpha_q])^{1/2} c|}. \tag{7}$$

In (5), (6), and (7) c depends upon α_p and the basic stability results are obtained by minimizing the right-hand sides in these inequalities. This requires the particular dependence of c on α_p be known so that the minimum may vary from problem to problem. However in the absence of this relationship the maximum value $\bar{c} = \rho(E)$ of c can be used to give stability results which will probably be more restrictive than the basic result (5), (6), or (7).

By minimizing the appropriate trignometric functions it immediately follows that the stability conditions for (2), (3) and (4) are respectively

$$\left| \frac{\bar{c}\, \Delta t}{\Delta x} \right| \leqslant \frac{1}{m^{1/2}}, \tag{8}$$

$$\left| \frac{\bar{c}\, \Delta t}{\Delta x} \right| \leqslant 1 \tag{9}$$

and

$$\left| \frac{\bar{c}\, \Delta t}{\Delta x} \right| \leqslant 1. \tag{10}$$

These conditions are also sufficient for stability provided reasonable assumptions, such as E having linear divisors, hold.

Since $E = \sum_{p=1}^{m} A_p l_p$ then, provided the matrices A_p are hermitian,

$$\bar{c}\,\frac{\Delta t}{\Delta x} \leqslant \sum_{p=1}^{m} \rho(A_p)\,|\,l_p\,|\,\frac{\Delta t}{\Delta x}.$$

The stability results above imply bounds on $|\,\bar{c}(\Delta t/\Delta x)|$. Given l_p it is possible to extend results (8), (9), and (10) in certain cases. In particular, with $m = 2$ in (3), it follows that $l_1 = \sin\alpha_1 \cos\alpha_2$, $l_2 = \sin\alpha_2 \cos\alpha_1$ so that

$$\left|\,\bar{c}\,\frac{\Delta t}{\Delta x}\,\right| \leqslant \frac{\Delta t}{\Delta x}\,|\sin\alpha_1\cos\alpha_2|\,\rho(A_1) + \frac{\Delta t}{\Delta x}\,|\sin\alpha_2\cos\alpha_1|\,\rho(A_2).$$

A sufficient condition for the right-hand side of this to be bounded by unity is that

$$\frac{\Delta t}{\Delta x}\,\rho(A_1) \leqslant 1, \qquad \frac{\Delta t}{\Delta x}\,\rho(A_2) \leqslant 1 \tag{11}$$

In a similar manner, $m = 3$ in (3) results in

$$\frac{\Delta t}{\Delta x}\,\rho(A_1 \text{ or } A_2 \text{ or } A_3) \leqslant \frac{3^{1/2}}{2}. \tag{12}$$

Abarbanel and Gottlieb [1] considered the matrices A_p mentioned above to be real and simultaneously symmetrizable. The stability results they obtained for (3) with $m = 2$ are identical to (11) and those for $m = 3$ are identical to (12). The matrices of the hydrodynamic equations can be simultaneously symmetrized, as shown by Turkel [3], so that if (3) was used in fluid flow problems the least restrictive stability condition would probably be (6).

3. *Comparison of Schemes*

The computational efficiency of a difference scheme depends upon its stability condition, the number of arithmetical operations required to advance the solution and the local truncation error. In any computation using (2), (3), or (4) to advance the solution of a problem in a region over a single time step, each scheme requires $\mathbf{F}_p(\mathbf{u})$, $p = 1, 2, \ldots, m$, to be evaluated at each mesh point in the region. Although a few extra additions are required to implement (4) compared with (3) and a few extra to implement (3) compared with (2), the contribution of these extra arithmetical operations to the total computation required to advance the solution over a single time step is minimal unless the vectors $\mathbf{F}_p(\mathbf{u})$ are of a very simple form. Thus the computational requirement of the three schemes are effectively the same, other factors being equal.

All three schemes have a local truncation error of the same order, these errors being functions of the third derivatives of $\mathbf{u}$. If these derivatives are all of the same sign, the error in (2) is less than that for (4) which in turn is less than that for (3). If the derivatives are not of the same sign it is impossible to compare these errors directly.

The stability results thus indicate that there is little to choose between the modified and averaging schemes and that both of these are probably more efficient than the basic scheme for finite m. The greater efficiency of (3) over (2) is in agreement with the results of Abarbanel and Gottlieb [1] for $m = 2$.

The rotated Richtmyer scheme uses the same averaging process as the modified Leap Frog scheme in approximating derivatives. It was shown in [2] that, in general, the rotated Richtmyer scheme was computationally more efficient than the basic Richtmyer scheme. It would thus appear that the use of some form of averaging process in approximating derivatives leads to increased computational efficiency.

4. *Suggested Extensions*

In (2) and (3) the space derivatives of (1) are approximated using mesh points not more than a mesh distance, in any coordinate direction, from the center of the domain of dependence. Equation (4) also satisfies these conditions. Other Leap Frog schemes using the same points configuration and which are a linear combination of (2) and (3) can be proposed. Equation (4) of course is the most obvious linear combination of (2) and (3). Since there would not be any improvement in either the stability condition or in the arithmetical requirements for such a scheme as compared with (4) the improvement would have to come from the truncation error. The ideal situation would be for a combination to give a higher order of accuracy. This is not possible. It is also not possible to select a combination which would always give a lower error than produced by (3) or (4). It would therefore hardly seem worthwhile to examine further Leap Frog approximations using the points configuration mentioned above.

References

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